Appendix B: Omitted Proofs and Simulation Results

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In this appendix we characterize the data generating process and required limits under Assumption 3 (Section B.1), the proof of Theorem A.1 (Section B.2), and omitted simulation results (Section B.3). Citations used only here are referenced at the end.

B.1 ASSUMPTION 3 Let \( \delta > 0 \) be a small number, and let \( \text{vec}(z) \) denote the vector that stacks columns of \( z \). Write \( \Gamma^*(\gamma) = \Gamma \times \cdots \times \Gamma \) the space of matrices \( \xi^{(-)} = [\xi^{(-)}_1, \ldots, \xi^{(-)}_k] \) or \( [\xi^{(-)}, \xi^{(+)}] \) (the dimension of \( \Gamma^*(\gamma) \) depends on the case). Let \( \hat{\xi} \) denote either \( \xi^{(-)}, \xi^{(+)} \) or \( \xi^{(-)}, \xi^{(+)} \). Similarly, \( \mathfrak{z} = \mathfrak{z}_1 \otimes \cdots \otimes \mathfrak{z}_k \) where \( \mathfrak{z}_i \) denote either \( \mathfrak{z}_i^{(+)} := \{ \arg\sup_{\gamma \in \Gamma} \{ (\partial/\partial \gamma) E[\epsilon_i F(\gamma' x_{t-1})] \} \} \), \( \mathfrak{z}_i^{(-)} := \{ \arg\inf_{\gamma \in \Gamma} \{ (\partial/\partial \gamma) E[\epsilon_i F(\gamma' x_{t-1})] \} \} \) or \( \mathfrak{z}_i^{(-)}, \mathfrak{z}_i^{(+)} \).

3.1: Each \( z_t \in \{ y_t, \hat{x}_{1,t}, \ldots, \hat{x}_{k-1,t} \} \) is \( L_2 \)-Near Epoch Dependent \( [\text{NED}] \) on a strong mixing base \( \{ \epsilon_t \} \) with mixing coefficients \( \alpha_t = O(i^{-\lambda}) \) for some \( \lambda > 1 \), in the following uniform sense (see Hill 2008, cf. Gallant and White 1988):

\[
\sup_{t \in \mathbb{Z}} \| z_t - E[z_t | \{ \epsilon_{t-i} \}_{i=0}^m] \|_r = O(v(m)) \quad \text{where} \quad v(m) \to 0 \quad \text{as} \quad m \to \infty.
\]

The error \( \epsilon_t \) has an almost everywhere positive continuous density, and \( \| \epsilon_t \|_{4+\delta} < \infty \).

Remark: We require \( \{ y_t, \hat{x}_t \} \) to be \( L_2 \)-NED on a strong mixing base \( \{ \epsilon_t \} \) in order to exploit weak limit theory in Hill (2008), cf. Bierens (1991). Mixing errors allow for GARCH and stochastic volatility errors (e.g. Carrasco and Chen 2002), any strong mixing \( \{ y_t, \hat{x}_t \} \) is automatically \( L_2 \)-NED on itself, and in general NED captures a broad array of linear and nonlinear time series with geometric or hyperbolic memory decay. Examples include threshold-type models (e.g. An and Huang 1996), and various nonlinear AR-GARCH (e.g. Meitz and Saikkonen 2008). See Hill (2011b) for a variety of examples and references in other contexts.

3.2: The parameter space \( \Phi \) is a compact subset of \( \mathbb{R}^p \). The known response function \( f_t(\phi) \) is \( \mathcal{G}_{t-1} \)-measurable, almost surely twice continuously differentiable on \( \Phi \).

3.3:
ii. \( A(\phi, \gamma, \xi) \to A(\phi, \gamma, \xi) \in \mathbb{R}^{(k+1) \times q} \) uniformly on \( \Phi \times \Gamma \times \Gamma^*(\gamma) \), where \( A(\phi, \gamma, \xi) = \)
\[\lim_{n \to \infty} \{(1/n) \sum_{i=1}^n w_i(\gamma, \xi) (\partial/\partial \phi') f_i(\phi) \times (H(\phi)^{-1} (\partial/\partial \phi') \bar{m}(\phi) \times \Xi)\} \quad \text{and} \quad H(\phi) = \lim_{n \to \infty} \{(\partial/\partial \phi') \bar{m}(\phi) \times \Xi \times (\partial/\partial \phi') \bar{m}(\phi)\}.

iii. There exists a unique element \( \xi^* \) of \( \mathcal{Z} \) satisfying \( \| \text{vec}(\hat{\xi}) - \text{vec}(\xi^*) \| = O_p(1/\sqrt{n}) \). Further,

\[
\sup_{\gamma \in \Gamma} \sup_{1 \leq t \leq n} \left| w_t(\gamma, \hat{\xi}) - w_t(\gamma, \xi^*) \right| = O_p(1/\sqrt{n}) \quad \text{and} \quad \sup_{\phi \in \Phi, \gamma \in \Gamma} \left| A(\phi, \gamma, \hat{\xi}) - A(\phi, \gamma, \xi^*) \right| = O_p(1/\sqrt{n})
\]

\[
\sup_{\gamma \in \Gamma} \sup_{1 \leq t \leq n} \left| g_t(\gamma, \hat{\xi}) - g_t(\gamma, \xi^*) \right| = O_p(1/\sqrt{n}) \quad \text{and} \quad \sup_{\phi \in \Phi} |H_n(\phi) - H(\phi)| = O_p(1/\sqrt{n}).
\]

3.4:

i. There exists a mapping \( \mu : \Gamma \times \Gamma^{(s)} \to \mathbb{R} \) satisfying \( (1/n) \sum_{t=1}^n u_t g_t(\gamma, \xi) \overset{P}{\to} \lim_{n \to \infty} (1/n) \sum_{t=1}^n E[u_t g_t(\gamma, \xi)] = \mu(\gamma, \xi) \) uniformly on \( \Gamma \times \Gamma^{(s)} \).

ii. There exists a mapping \( \Sigma : \Gamma \times \Gamma \to \mathbb{R}^{(k+1) \times (k+1)} \) satisfying \( (1/n) \sum_{t=1}^n E[\xi_t^2 \gamma_{t,1}^{-1}] g_t(\gamma_1, \xi) g_t(\gamma_2, \xi) \overset{P}{\to} \Sigma(\gamma_1, \gamma_2) \) and \( (1/n) \sum_{t=1}^n E[\xi_t^2 g_t(\gamma_1, \xi) g_t(\gamma_2, \xi)'] \to \Sigma(\gamma_1, \gamma_2) \) uniformly on \( \Gamma \times \Gamma \).

iii. \( \limsup_{n \to \infty} \sup_{\gamma \in \Gamma} 1/n \sum_{i=1}^n E[\xi_t^2 u_t g_t(\gamma, \xi)]^{2+\delta} < \infty \).

3.5: Uniformly on \( 1 \leq t \leq n \):

\[\sup_{\gamma \in \Gamma, \xi \in \Gamma^{(s)}} |g_t(\gamma, \xi)| \ , \ |g_t(\gamma, \xi)'| < K, \quad \sup_{\gamma \in \Gamma, \xi \in \Gamma^{(s)}} |(\partial/\partial \gamma_t) g_t(\gamma, \xi)| \ , \ |(\partial/\partial \gamma_t) g_t(\gamma, \xi)'| < K, \quad \sup_{\gamma \in \Gamma, \xi \in \Gamma^{(s)}} |(\partial/\partial \phi_t) c_t(\phi)| \ , \ |(\partial/\partial \phi_t) c_t(\phi)'| < C. \]

3.6: The local alternative random variable \( u_t \) is \( \mathcal{Z}_{t-1} \)-measurable, governed by a non-generate distribution.

### B.2 PROOF OF THEOREM A.1

Recall the claim.

**THEOREM A.1.** Assume Assumptions 1 and 2, and \( P(E[\epsilon_t | \mathcal{Z}_{t-1}] = 0) < 1 \) hold. Then \( S = \{0\} \) if and only if \( \epsilon_t \) is \( L_2 \)-orthogonal to the closed linear span of \( \{x_{i,t} F'(\xi'_{(i)} x_t)\}_{i=1}^k \), and otherwise \( S \) is empty.

We only treat \( \xi_{(i)} = \xi_{(i)}^{(s)} \), and for the sake of convention assume

\[
\frac{\partial}{\partial \gamma_t} E[\epsilon_t F'(\gamma' x_t)] |_{\gamma = \xi_{(i)}} \geq 0.
\]

All subsequent results carry over to the general case \( (\partial/\partial \gamma_t) E[\epsilon_t F'(\gamma' x_t)] |_{\gamma = \xi_{(i)}} \geq 0 \), and to arginf.\( \gamma \in \Gamma \{(\partial/\partial \gamma_t) E[\epsilon_t F'(\gamma' x_t)]\} \).

The proof requires one supporting result. Define

\[
\varpi(\gamma, \xi) := E \left[ \epsilon_t \left( F'(\gamma' x_t) - \sum_{i=1}^k \gamma_i x_{i,t} F'(\xi'_{(i)} x_t) \right) \right].
\]

**LEMMA B.1.** Under Assumptions 1 and 2 if \( P(E[\epsilon_t | \mathcal{Z}_{t-1}] = 0) < 1 \) then \( \varpi(\gamma, \xi) = 0 \) for \( \gamma \in \Gamma \) if and only if \( \gamma = 0 \).

**Remark:** The two structures \( E[\epsilon_t F'(\gamma' x_t)] \) and \( \sum_{i=1}^k \gamma_i E[\epsilon_t x_{i,t} F'(\xi'_{(i)} x_t)] \) are equal only at the origin \( \gamma = 0 \). But this means if \( E[\epsilon_t F'(\gamma' x_t)] = 0 \) under \( H_1 \) for any \( \gamma \neq 0 \) then \( \sum_{i=1}^k \gamma_i E[\epsilon_t x_{i,t} F'(\xi'_{(i)} x_t)] 
eq 0 \) hence at least one moment condition \( E[\epsilon_t x_{i,t} F'(\xi'_{(i)} x_t)] 
eq 0 \). Similarly, if all moments \( E[\epsilon_t x_{i,t} F'(\xi'_{(i)} x_t)] = 0 \) then \( E[\epsilon_t F'(\gamma' x_t)] = 0 \) for all \( \gamma \neq 0 \).

**PROOF OF THEOREM A.1.** Assume \( P(E[\epsilon_t | \mathcal{Z}_{t-1}] = 0) < 1 \). By Lemma B.1 we know for each \( \gamma \neq 0 \)

\[
\varpi(\gamma, \xi) = E \left[ \epsilon_t F'(\gamma' x_t) - \sum_{i=1}^k \gamma_i E \left[ \epsilon_t x_{i,t} F'(\xi'_{(i)} x_t) \right] \right] \neq 0.
\]
Trivially, therefore, at least one moment condition $E[\epsilon_t F(\gamma' x_t)], E[\epsilon_t x_{1,t} F'(\xi'_1x_t)], ..., \text{ or } E[\epsilon_t x_{k,t} F'(\xi'_kx_t)]$ must be non-zero, hence $E[\epsilon_t w_t(\gamma, \xi)] \neq 0$ for every $\gamma \neq 0$.

Finally, under Assumption 1

$$E[\epsilon_t w_t(0, \xi)] = \left[0, E \left[ \epsilon_t x_{1,t} F' \left( \xi'_1x_t \right) \right], ..., E \left[ \epsilon_t x_{k,t} F' \left( \xi'_kx_t \right) \right] \right],$$

hence $E[\epsilon_t w_t(0, \xi)] = 0$ if and only if $\epsilon_t$ is orthogonal to the closed linear span of $\{x_{i,t} F'(\xi'_i x_t)\}_{i=1}^k$. \(\square\)

In order to prove Lemma B.1 we require an easy extension of Theorem 1 of Bierens and Ploberger (1997) and Theorem 2.3 and Corollary 3.9 of Stinchcombe and White (1998). Let $h_t : D \rightarrow \mathbb{R}^k$ be an $\mathcal{F}_{t-1}$-measurable, uniformly bounded function, where $D$ is an arbitrary subset of $\mathbb{R}^l$ for some $l \geq 0$. Write $h_t = h_t(\delta)$ by convention when $l = 0$. Examples of $h_t(\delta)$ include $x_t$, $x_t^{\delta} \times \text{sign}(x_t)$ and $(\partial \phi / \partial \phi) f_t(\phi)$ where $\delta = \phi$ provided $x_t$ and $(\partial \phi / \partial \phi) f_t(\phi)$ are bounded with probability one.

**LEMMA B.2.** Let weight $F$ satisfy Assumption 1. If $P(E[\epsilon_t | \mathcal{F}_{t-1}] = 0) < 1$ then for each $\delta \in D$ the set $\mathcal{S} = \bigcap_{\delta=1}^k \{ \gamma \in \mathbb{R}^k : E[\epsilon_t h_{i,t}(\delta) F(\gamma' x_t)] = 0 \}$ and $P(\gamma' x_t \in \mathcal{R}_0) = 1 \}$ has Lebesgue measure zero and is nowhere dense in $\mathbb{R}^k$.

**Remark:** By Assumption 1 and Corollary 3.9 of Stinchcombe and White (1998), Lemma B.2 holds with $F(\gamma)$ replaced by $F'(.).$ If $x_t$ is not bounded with probability one then replace it with any Borel measurable, bounded one-to-one mapping $\Psi(x_t)$ (Bierens 1990).

**PROOF OF LEMMA B.1.** Recall

$$\xi_{(i)} \in \mathcal{S}_i := \left\{ \arg\sup_{\gamma \in \Gamma} \left\{ \frac{\partial}{\partial \gamma_i} \mu(\gamma) \right\} \right\} = \left\{ \arg\sup_{\gamma \in \Gamma} \left\{ E \left[ \epsilon_t \frac{\partial}{\partial \gamma_i} F(\gamma' x_t) \right] \right\} \right\}. $$

The construction of $\xi_{(i)}$ implies for all $\gamma \in \Gamma$

$$E \left[ \epsilon_t x_{i,t} F'(\gamma' x_t) \right] \leq E \left[ \epsilon_t x_{i,t} F' \left( \xi'_{(i)} x_t \right) \right] = \sup_{\gamma \in \Gamma} E \left[ \epsilon_t x_{i,t} F'(\gamma' x_t) \right]$$

and Assumptions 1 and 3 imply $\varpi(0, \xi) = E[\epsilon_t F(0' x_t)] = 0$. Differentiate $\varpi(\gamma, \xi)$ with respect to $\gamma_j$, and add and subtract $E[\epsilon_t x_{j,t} F'(0' x_t)]:$

$$\frac{\partial}{\partial \gamma_j} \varpi(\gamma, \xi) = E \left[ \epsilon_t x_{j,t} \{ F'(\gamma' x_t) - F'(0' x_t) \} \right]$$

$$= - E \left[ \epsilon_t x_{j,t} \{ F' \left( \xi'_{(j)} x_t \right) - F'(0' x_t) \} \right] \leq 0.$$ 

Thus $\varpi(\gamma, \xi)$ is zero at $\gamma = 0$ and is weakly decreasing in $\gamma$, and (1) implies $E[\epsilon_t x_{j,t} \{ F'(\xi'_{(j)} x_t) - F'(0' x_t) \}] \geq 0 \ \forall j = 1..k$.

In order to prove weak inequality (2) is in fact strict, there are two cases.

**Case 1** ($E[\epsilon_t x_{j,t} \{ F'(\xi'_{(j)} x_t) - F'(0' x_t) \}] = 0$): Trivially

$$E \left[ \epsilon_t x_{j,t} \{ F'(\gamma' x_t) - F'(0' x_t) \} \right]_{\gamma=0} = 0.$$ 

Lemma B.2 therefore implies there exists an open neighborhood $N(0) \subset \Gamma$ of zero satisfying

$$E \left[ \epsilon_t x_{j,t} \{ F'(\gamma' x_t) - F'(0' x_t) \} \right] \neq 0 \ \forall \gamma \in N(0)/0.$$ 

Since by assumption $E[\epsilon_t x_{j,t} \{ F'(\xi'_{(j)} x_t) - F'(0' x_t) \}] = 0$ we deduce from (2)

$$E \left[ \epsilon_t x_{j,t} \{ F'(\gamma' x_t) - F'(0' x_t) \} \right] < 0 \ \forall \gamma \in N(0)/0.$$
Thus \(\varpi(\gamma, \xi)\) is zero at \(\gamma = 0\), strictly decreasing arbitrarily close to \(\gamma = 0\), and weakly decreasing everywhere else. This implies \(\varpi(\gamma, \xi) \neq 0\) for every \(\gamma \neq 0\).

**Case 2** \(\left(\mathbb{E}[\epsilon_{i,x_1,t}\{F'((\xi_0')x_t) - F'(0'x_t)] > 0\right): \) Use (2) to deduce for each \(j = 1...k\)

\[
\frac{\partial}{\partial \gamma_j} \varpi(\gamma, \xi)|_{\gamma = 0} = 0 - \mathbb{E}\left[\epsilon_{i,x_1,t}\left\{F'\left((\xi_0')x_t\right) - F'(0'x_t)\right\}\right] < 0.
\]

Again, \(\varpi(\gamma, \xi)\) is zero at \(\gamma = 0\), strictly decreasing at \(\gamma = 0\) and weakly decreasing everywhere else. \( \Box \).

### B.3 OMITTED SIMULATION RESULTS

Finally, we present omitted simulation results for fixed CM tests.

#### Table B.1: Fixed CM Tests Rejection Frequencies

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<th>ESTAR</th>
<th>SETAR</th>
<th>BILIN</th>
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a. The nuisance parameter is fixed at the mid-point \(\gamma_0 = [5.25, ..., 5.25]^T\) of \(\Gamma = [5, 10]^p+1\).
b. \(L = \text{logistic; } E = \text{exponential}\). Values are rejection frequency at the 5\% level.
c. Bierens’ (1990) and Lee et al’s (1996) CM test with logistic or exponential weight \(F(\gamma'; \psi_1)\).
d. Hill’s (2008) STAR test with logistic or exponential weight \(x_tF(\gamma'; \psi_1)\).
e. The Uniformly Most Powerful test. Each test statistic is designed to be UMP for the particular \(H_1\).

In simulations not reported here, each statistic obtains empirical size roughly equal to nominal size.
REFERENCES
